3.3 Alternating Series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

where each a_n is positive.

3.3A Theorem:

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that a) $a_1 \geq a_2 \geq a_3 \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and

b) $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Proof

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots \infty$$

$s_1 = a_1$

$s_2 = a_1 - a_2$

$s_3 = a_1 - a_2 + a_3$

$s_4 = a_1 - a_2 + a_3 - a_4$

$s_5 = a_1 - a_2 + a_3 - a_4 + a_5$

$\vdots$

$s_{2n-1} = a_1 - a_2 + a_3 - \cdots + a_{2n-1}$

$s_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n}$

Consider first the partial sums with odd index

$s_1, s_3, s_5, \cdots$

given $a_2 \geq a_3$

$\Rightarrow a_3 - a_2 \leq 0$

$s_3 = a_1 - a_2 + a_3 = s_1 + \underbrace{(a_3 - a_2)}_{\leq 0}$

$\Rightarrow s_3 \leq s_1$

$a_4 \geq a_5$

Also $s_5 = a_1 - a_2 + a_3 - a_4 + a_5 = s_3 + \underbrace{(a_5 - a_4)}_{\leq 0}$

$a_5 - a_4 \leq 0$

$s_5 \leq s_3$

$\Rightarrow s_1 \geq s_3 \geq s_5 \geq \cdots \geq s_{2n-1} \geq \cdots$

$\Rightarrow$ The sequence $\{s_{2n-1}\}_{n=1}^{\infty}$ is decreasing sequence.

$s_{2n-1} = s_1 - s_2 + s_3 - s_4 - \cdots + s_{2n-1}$

$s_{2n-1} = \underbrace{(s_1 - s_2)}_{+ve} + \underbrace{(s_3 - s_4)}_{+ve} + - \cdots + \underbrace{s_{2n-1}}_{+ve}$ $\quad s_1 = a_1 \geq 0$

$s_{2n-1} \geq 0 \quad \forall n$ bounded below $\quad s_3 = a_1 - a_2 + a_3$

$\Rightarrow \{s_{2n-1}\}_{n=1}^{\infty}$ is decreasing and bounded below $= \underbrace{(a_1 - a_2)}_{\geq 0} + \underbrace{a_3}_{\geq 0}$

$\therefore \{s_{2n-1}\}_{n=1}^{\infty}$ is convergent

say $\lim\limits_{n\to\infty} s_{2n-1} = L$ —— (1)

$a_1 \geq a_2$

$s_2 = a_1 - a_2 \qquad a_1 - a_2 \geq 0$

$s_4 = \cancel{s}\, a_1 - a_2 + a_3 - a_4 \qquad a_3 \geq a_4$

$s_4 = s_2 + \underbrace{(a_3 - a_4)}_{\geq 0} \qquad a_3 - a_4 \geq 0$

$\Rightarrow s_4 \geq s_2 \quad$ ||ly $s_6 = s_4 + \underbrace{(a_5 - a_6)}_{\geq 0} \quad s_6 \geq s_4$,

$\Rightarrow s_2 \leq s_4 \leq s_6 \leq - \cdot \leq s_{2n} \leq - -$.

$\Rightarrow$ The sequence $\{s_{2n}\}_{n=1}^{\infty}$ is increasing sequence,

$s_2 = a_1 - a_2 \qquad s_2 \leq a_1$

$a_2 \geq a_3$

$s_4 = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - a_4 \qquad a_2 - a_3 \geq 0$

$s_4 \leq a_1$

$\forall n \quad s_{2n} \leq a_1$

$\Rightarrow \{s_{2n}\}_{n=1}^{\infty}$ is bounded above

$\Rightarrow \{s_{2n}\}_{n=1}^{\infty}$ increasing seq and bounded ~~below~~ above

$\Rightarrow \{s_{2n}\}_{n=1}^{\infty}$ is convergent sequence

say $\lim\limits_{n\to\infty} s_{2n} = M$

$$\lim_{n\to\infty} s_{2n} - s_{2n-1} = L-M$$

$$\lim_{n\to\infty} a_{2n} = L-M$$

But given $\lim\limits_{n\to\infty} a_n = 0 \Rightarrow \lim\limits_{n\to\infty} a_{2n} = 0$

$$L-M = 0$$
$$L = M$$

$$\Rightarrow \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1} = L$$

$$\Rightarrow \lim_{n\to\infty} s_n = L$$

$\Rightarrow$ The sequence $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

3.4 Conditional Convergence and absolute Convergence.

3.4A Defn: Let $\sum\limits_{n=1}^{\infty} a_n$ be a Series of real numbers.

a) If $\sum\limits_{n=1}^{\infty} |a_n|$ Converges, we say that $\sum\limits_{n=1}^{\infty} a_n$ Converges absolutely.

b) If $\sum\limits_{n=1}^{\infty} a_n$ converges but $\sum\limits_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum\limits_{n=1}^{\infty} a_n$ converges conditionally.

2.4B Theorem:

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ Converges.

To prove $\sum_{n=1}^{\infty} a_n$ Converges.

(ie) To prove the sequence $\{s_n\}_{n=1}^{\infty}$ is converges

where $s_n = a_1 + a_2 + \cdots + a_n$

It is enough to prove $\{s_n\}_{n=1}^{\infty}$ is Cauchy Sequence.

[Since every Cauchy sequence is convergent sequence]

Given $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ie) $\sum_{n=1}^{\infty} |a_n|$ converges.

Let $t_n = |a_1| + |a_2| + \cdots + |a_n|$ is converges.

(ie) The sequence $\{t_n\}_{n=1}^{\infty}$ is Cauchy sequence.

Since every convergent sequence is Cauchy sequence.

$\therefore$ seq $\{t_n\}_{n=1}^{\infty}$ is Cauchy Sequence.

By defn given $\varepsilon > 0$ there exists $N \in I$ such that

$|t_m - t_n| < \varepsilon \quad (\forall\, m, n \geq N)$

But (if $m > n$, say)

$$|t_m - t_n| = \left| (|a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|) - (|a_1| + |a_2| + \cdots + |a_n|) \right|$$

$$= \left| |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| \right| < \varepsilon$$

$$|t_m - t_n| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon \quad \text{--- (1)}$$

now $|s_m - s_n| = |a_{n+1} + a_{n+2} + \cdots + a_m|$

$\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|$

using ①

$|s_m - s_n| < \epsilon \quad \forall\, m, n \geq N$

$\Rightarrow \{s_n\}_{n=1}^{\infty}$ is a Cauchy Sequence.

$\Rightarrow \{s_n\}_{n=1}^{\infty}$ is a convergent sequence

$\Rightarrow$ The Series $\sum_{n=1}^{\infty} a_n$ is convergent Sequence.

3.6 Tests for absolute Convergence.

3.6A Definition Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers. we shall say that $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ if there exists $N \in I$ such that

$$|a_n| \leq |b_n| \quad n \geq N.$$

In this case we write

$$\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n.$$

3.6B Theorem: If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$, where $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ also converges absolutely

(or)

[If $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |b_n| < \infty$, then $\sum_{n=1}^{\infty} |a_n| < \infty$

Proof: Given $\sum_{n=1}^{\infty} b_n$ converges ~~also~~ absolutely

$\Rightarrow \sum_{n=1}^{\infty} |b_n|$ is converges.